Exercises

Exercise 1 *Prove Theorem 2.2. (Hint: write the FOC of the problem and differentiate its right-hand side with respect to* α*. Then, show that the resulting expression is decreasing in* α*).*

Exercise 2 *Consider a random payoff* $x = (h, -h)'$ *whose uncertainty is thus resolved at two diferent states of nature with* $h > 0$*. For a given level of wealth* Y*, denote by* $\pi(Y, h)$ *the probability at state 1 that would make an investor with utility* u *indifferent between purchasing or not* x *at no charge. That is,* $\pi(Y, h)$ *is the probability for which it holds that*

$$u(Y) = \pi(Y, h)\, u(Y + h) + [1 - \pi(Y, h)]\, u(Y - h).$$

Use a second-order Taylor approximation of $u(Y + h)$ *and* $u(Y - h)$ *around* $h = 0$ *to show that*

$$\pi(Y, h) - \frac{1}{2} \simeq \frac{1}{4} h \mathcal{A}(Y).$$

Exercise 3 *As in the lecture notes, let* $A = \mathbf{E}'\mathbf{\Sigma}^{-1}\mathbf{E}$, $B = \mathbf{E}'\mathbf{\Sigma}^{-1}\mathbf{1}$ *and* $C = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}$*, where* $\mathbf{\Sigma}$ *is the covariance matrix of the basis returns and* $\mathbf{E}$ *is a vector with their corresponding excess returns. Show that a frontier return* R *is efficient (in the sense that there does not exist another portfolio with a higher mean but the same variance) if its expected return* $E(R) > \frac{B}{C}$*.*

Exercise 4 *Let* R^a *be a mean-variance return and let* R^b *be any other return which has the same mean as* R^a*. Show that* $cov(R^a, R^b) = Var(R^a)$*.*

Exercise 5 *Prove the following statements:*

a. *In the expected return-variance space, the line passing between the mean-variance* R^a *and the global minimum variance portfolio intersects the expected return axis at* $E(R^{za})$*. (hint: in mean-variance space find the expression for the intercept of the line connecting* R^a *and the global minimun variance return; then apply Lemma 2.7).*

b. *In the return-standard deviation space, the tangent to the mean-variance frontier at* R^a *intersects the expected return axis at* $E(R^{za})$ *(hint: find the slope of the function that gives the standard deviation as a function of the mean for mean-variance returns and the slope of the above tangent and set them equal. Then factor out the value of the intercept of the tangent and apply Lemma 2.7).*

c. *If there is a risk-free asset any two returns in the mean-variance frontier are perfectly correlated.*

Exercise 6 *Prove Lemma 2.7 (hint: it will help to look at the arguments of the proof of Theorem 2.10).*

Exercise 7 *Show the following results that involve the HARA class of utility functions:*

a. *The utility function (negative exponential utility)*

$$u(c) = -\exp(-Ac)$$

displays constant absolute risk aversion equal to A.

b. *The utility function (power utility)*

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \qquad \gamma > 0 \tag{1}$$

displays constant relative risk aversion equal to γ.

c. *The elasticity of intertemporal substitution is defined as*

$$-\frac{IMRS}{c_1/c_0}\frac{\partial(c_1/c_0)}{\partial IMRS}.$$

Show that (1) displays constant elasticity of intertemporal substitution equal to $1/\gamma$.

Exercise 8 *Consider the problem*

$$\max_w E\left[-\exp\left(-Aw'\mathbf{R}\right)\right] \tag{2}$$
$$s.t. \quad w'\mathbf{R} = 1$$

and the set

$$\underline{R}_e = \left\{R_e : R_e \in \underline{R} \text{ and } R_e \text{ solves (2) for some } A \in \Re\right\}.$$

Assume there is a risk-free asset and the problem has an interior solution for $A = 1$.

a. *Show that a return $R_e \in \underline{R}_e$ if and only if it is of the form*

$$R_e = \alpha R^f + (1-\alpha) R_e^*$$

where R_e^* is the return that solves (2) for $A = 1$.

b. *Show that if R_e belongs to both $\underline{R}$ and $\underline{R}_e$, then*

$$E(R) - R^f = \frac{cov\left[\exp(-AR_e), R\right]}{cov\left[\exp(-AR_e), R_e\right]}\left[E(R_e) - R^f\right], \qquad \forall R \in \underline{R}$$

for some $A \neq 0$.

c. *Consider a two-period economy where agents indicate their preferences with time-additive total utility and period utility of the form*

$$u^k(c) = -\exp(-A_k c), \qquad k = 1, \ldots, K.$$

Show that in a financial equilibrium it must hold that

$$m_k = \frac{\exp(-R_e^*)}{E[\exp(-R_e^*)]\, R^f}, \qquad k = 1, \ldots, K.$$

where m_k denotes agent k's IMRS.

d. *Show that in the above economy the same equilibrium prices are attained by means of a representative investor who owns the economy's total wealth and whose period utility function is given by*

$$u(c) = -\exp\left(-A_m^{-1} c\right)$$

where $A_m \equiv \sum_{k=1}^{K} A_k^{-1}$.

Exercise 9 *Prove Theorem 2.21 (hint: use similar arguments to the ones in the proof of Theorem 1.6).*

Exercise 10 *Let $\underline{X}$ be the set of attainable payoffs spanned by a set of basis assets with strictly positive price which contains a risk-free asset.*

a. *Show that if the canonical portfolio problem with negative exponential utility has a solution for some given initial wealth Y_0, then there exists a payoff x_f^* such that*

$$E\left[\exp\left(-x_f^*\right) x\right] = p(x) \qquad \forall x \in \underline{X}.$$

b. *Show that if x_f^* exists, then*

$$-E(m \ln m) \geq E\left[\exp\left(-x_f^*\right) x_f^*\right]$$

for all SDF m.

Remark 11 *Stutzer (1995) uses this result in order to develop what he calls information bound on the set of SDF's. This is a diagnostic tool for testing asset pricing models in the lines of Hansen and Jagannathan bounds.*

Exercise 12 *Consider the model*

$$m = b' f$$

where $b = [b_1, \ldots, b_J]'$ is a vector of constants and $f = [f_1, \ldots, f_J]'$ is a vector of J factors. For every $j = 1, \ldots, J$ define

$$\widehat{f}_j \equiv proj\left(f_j \mid \underline{X}\right) = \widehat{X}' E\left[\widehat{X}\widehat{X}'\right]^{-1} E\left[\widehat{X} f_j\right]$$

where $\widehat{X}$ is the payoff matrix that does not include redundant payoffs, that is, it is a matrix that contains a subset of linearly independent payoffs that span the set of attainable payoffs $\underline{X}$. Thus, $\widehat{f}_j$ is the projection of factor f_j onto the space $\underline{X}$. Show that the vector $\widehat{f} = \left[\widehat{f}_1, \ldots, \widehat{f}_J\right]'$ carries the same pricing implications as f in the sense that if

$$E(mx) = p(x), \qquad \forall x \in \underline{X}$$

then

$$E(\widehat{m}x) = p(x), \qquad \forall x \in \underline{X}$$

where $\widehat{m} = b'\widehat{f}$.

Remark 13 *These factors are called "factor-mimicking portfolios" because they mimic the behavior of the original factors f. These constructs rely on the same idea that gives x^*: we project m onto $\underline{X}$ to find x^* which prices payoffs as any other stochastic discount factor. We obtain the factor-mimicking portfolios by projecting each individual factor onto the set of attainable payoffs and we find a factor model with payoffs as factors which carries all the pricing implications of the initial sdf. Once the underlying macroeconomic factors f that contain the sources of risk are detected, it is usually a good idea to find their mimicking portfolios $\widehat{f}$. This is because high-frequency data on $\widehat{f}$ free of measurement errors is easily available even on a minute-by-minute basis (recall they are themselves payoffs) which is not at all the case of the original factors f.*